1.a

$$H(f) = -j\,\mathrm{sgn}(f) \overset{\mathcal{F}^{-1}}{\rightarrow} \frac{1}{\pi t} = h(t)$$

$\boxed{\text{It's a real system because the impulse response is real.}}$

For any LTI system with impulse response $h(t)$, the response $y(t)$ is a convolution of the input x and h. y is real if $y = y^*$. Assume x is real.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = y^*(t) = \left(\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right)^* = \int_{-\infty}^{\infty} h^*(\tau)\, x^*(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h^*(\tau)\, x(t-\tau)\, d\tau \quad (x \text{ is assumed to be real})$$

$\rightarrow \quad h^*(t) = h(t)$ $\quad$ h being real is sufficient for the system to be real.

1.b

A system is causal if an output at some time t_0 depends only on inputs prior to t_0. ($y(t_0)$ depends only on $x(t)$ for $t \le t_0$). A simple counter example is to compare the impulse response to the zero response for $t_0 = -1$. Any linear system will give zero output with zero input. For $t_0 = -1$, $\delta(t) = 0$ for all $t \le t_0$, so $h(-1)$ (the output given $\delta(t)$ as input) should be 0, but it is $\frac{-1}{\pi}$. so the system is non causal.

In general:

$y(t_0)$ should be equal if the input is $x(t)$ or $x(t)\,u(t_0 - t)$, because these inputs are equal for $t \le t_0$.

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t_0-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, u(t_0-\tau)\, h(t_0-\tau)\, d\tau \quad \text{(by assumption)}$$

$$= \int_{-\infty}^{\infty} x(t_0-\lambda)\, \underbrace{h(\lambda)\, u(\lambda)}_{h_c}\, d\lambda \quad (\lambda = t_0 - \tau)$$

If $h(t) = h_c(t) (= h(t)\,u(t))$ then the system is causal. If $h(t) \ne h(t)u(t)$, then there must be some input that leads to a non-causal output.

1.c)

A system is BIBO stable if any bounded input gives a bounded output.

An obvious counterexample is $u(t)$:

$$y(0) = \int_{-\infty}^{\infty} u(0-\tau)\frac{1}{\pi\tau}\,d\tau = \int_{-\infty}^{0} \frac{1}{\pi\tau}\,d\tau = -\infty$$

In general, if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$, then the system is BIBO stable.

Let $|x(t)| \le M$ for all t.

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau\right| \le \int_{-\infty}^{\infty} |x(t-\tau)|\,|h(\tau)|\,d\tau \le \int_{-\infty}^{\infty} M\,|h(\tau)|\,d\tau$$

$$= M\int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

M is finite, so the integral must be as well if the output is to be bounded.

2.a.)

$$\mathcal{F}\{\hat{x}(t)\} = \hat{X}(f) = X(f)\cdot(-j\,\mathrm{sgn}(f))$$

the ESD of $X(f)$ is $|X(f)|^2$, so the ESD of $\hat{X}(f)$ is

$$|\hat{X}(f)|^2 = |X(f)(-j\,\mathrm{sgn}(f))|^2 = |X(f)|^2\,|-j\,\mathrm{sgn}(f)|^2 = |X(f)|^2\cdot 1 = |X(f)|^2$$

2.b)

$$\langle \hat{x}, x\rangle = \int_{-\infty}^{\infty} \hat{x}(t)\,x^*(t)\,dt = \int_{-\infty}^{\infty} \hat{X}(f)\,X^*(f)\,df \quad \text{(parseval)}$$

$$= \int_{-\infty}^{\infty} X(f)(-j\,\mathrm{sgn}(f))\,X^*(f)\,df = -j\int_{-\infty}^{\infty} \mathrm{sgn}(f)\,|X(f)|^2\,df$$

$|X(f)|^2$ is even: $X(f) = X^*(-f)$ (X is hermitian because x is real)

$$|X(f)|^2 = X(f)\,X^*(f) = X^*(-f)\,X^*(f) = X(-f)\,X(f)$$

absolute value is always real

$$|X(-f)|^2 = X(-f)\,X^*(-f) = X(f)\,X(f) = |X(f)|^2 \quad \underline{\text{even}}$$

So we're integrating an even function times an odd one ($\mathrm{sgn}(f)$ is odd), which gives an odd function. Integrated over a symmetric interval, this gives <u>zero</u>.

There's probably a more straightforward way to show this, but this is what I came up with.

3.a

HW4

$H(f) = -j\,\mathrm{sgn}(f)$

The terms of the fourier series are eigen signals, so:

$$X[n]\, e^{2\pi j f_0 n t} \xrightarrow{H} H(nf_0)\, X[n]\, e^{2\pi j f_0 n t} = \hat{X}[n]\, e^{2\pi j f_0 n t}.$$

$$\hat{X}[n] = H(nf_0)\, X[n] = \boxed{-j\,\mathrm{sgn}(nf_0)\, X[n]}$$

$$= \boxed{\begin{cases} -j\, X[n] & n > 0 \\ +j\, X[n] & n < 0 \\ 0 & n = 0 \end{cases}}$$

3.b

Given FS coefficients $X[n]$ the FS coeffs. of the PSD are $\boxed{|X[n]|^2}$.

As a continuous frequency spectrum, $S_x(f) = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} |X[n]|^2 e^{2\pi j f_0 n t} \right\}$

$$= \boxed{\sum_{n=-\infty}^{\infty} |X[n]|^2\, \delta(f - nf_0)}$$

The book kind of glosses over this, and I don't think I've covered it in as much detail as I should have. So here it is:

$$\rho_x(\tau) = \lim_{W\to\infty} \frac{1}{W} \int_{-W/2}^{W/2} x(t)\, x^*(t-\tau)\, dt$$ for any power signal x, this is the auto correlation.

If $x(t)$ is T-periodic, then so is $\rho_x(\tau)$:

$$\rho_x(\tau + T) = \lim_{W\to\infty} \frac{1}{W} \int_{-W/2}^{W/2} x(t)\, x^*(t-\tau-T)\, dt$$

$$= \lim_{W\to\infty} \frac{1}{W} \int_{-W/2}^{W/2} x(t)\, x^*(t-\tau)\, dt \qquad (x(t) = x(t+T))$$

$$= \rho_x(\tau)$$

Clearly, $\rho_x(\tau)$ has a Fourier series. Let's find the coefficients.

$$\rho_x(t) = \frac{1}{T} \int_0^T x(t)\, x^*(t-\tau)\, dt = \frac{1}{T} \int_0^T x(t)\, x^*(-(\tau - t))\, dt$$

This is the cyclic convolution of $x(t)$ and $x^*(-t) = x^\dagger(t)$. It's not hard to show that the Fourier series coefficients of the convolution are $X[n]\, X^*[n] = |X[n]|^2$

continued...

3.b cont.

$$y(t) = \frac{1}{T}\int_0^T x(\lambda)\, h(t-\lambda)\, d\lambda = \frac{1}{T}\int_0^T x(\lambda) \sum_{n=-\infty}^{\infty} H[n]\, e^{2\pi j n (t-\lambda)/T}\, d\lambda$$

$$= \sum_{n=-\infty}^{\infty} H[n] \left(\frac{1}{T}\int_0^T x(\lambda)\, e^{-2\pi j n \lambda/T}\, d\lambda\right) e^{2\pi j n t/T}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{H[n]\, X[n]}_{Y[n]}\, e^{2\pi j n t/T}$$

$$x(t) \xrightarrow{\mathcal{F}} X[n] \quad \text{and} \quad x^{*}(t) \xrightarrow{\mathcal{F}} X^{*}[n]$$

so $\rho_x(t) \xrightarrow{\mathcal{F}} |X[n]|^2$

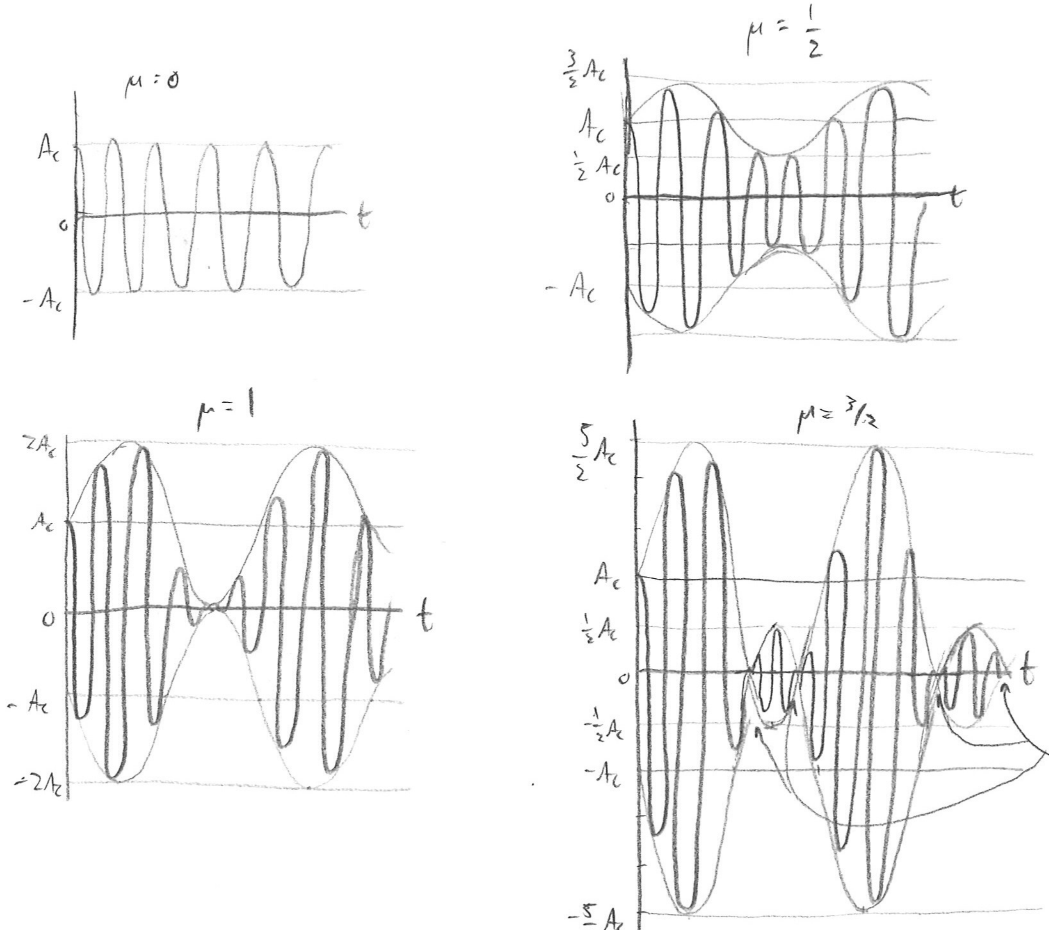
Anyway, it's enough to just look at the book and say $PSD = |X[n]|^2$.

or $PSD = \sum_{n=-\infty}^{\infty} |X[n]|^2 \delta(f - n f_0)$

3.c

As before, $S_y(f) = |H(f)|^2 S_x(f)$. Since $|H(f)|^2 = |-j\,\mathrm{sgn}(f)|^2 = 1$, $\boxed{S_y(f) = S_x(f)}$

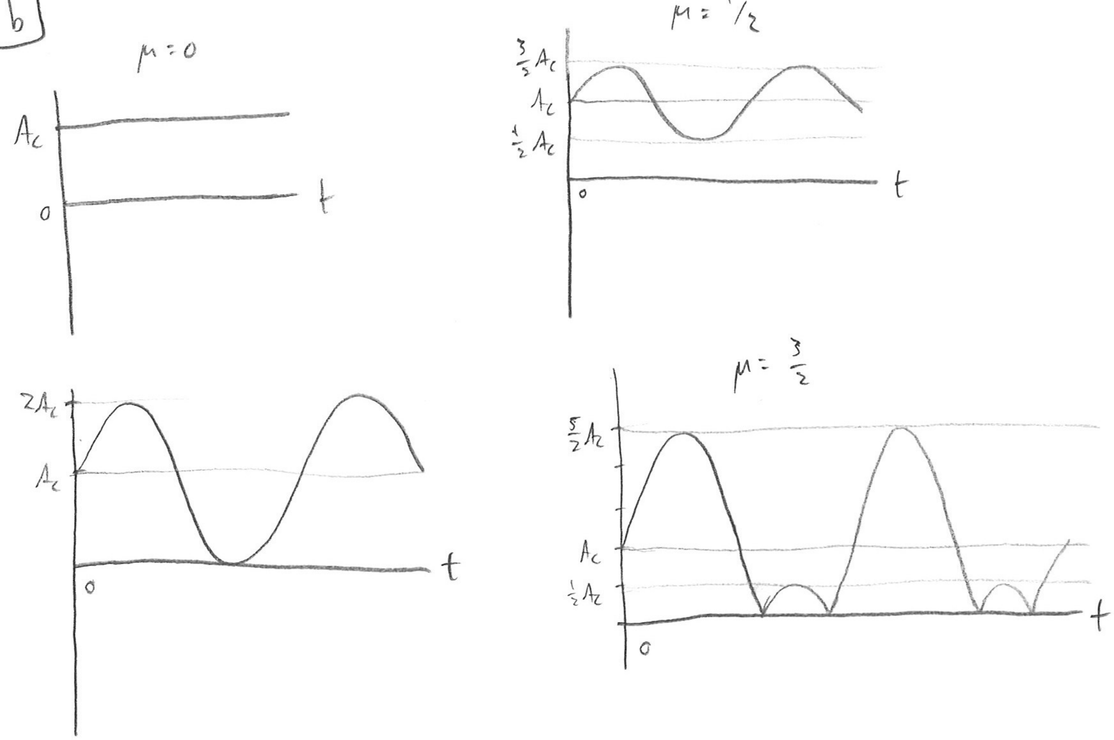
4.a



It's hard to draw, but there are phase reversals here

4.b

HW 4



Really these might be scaled by some factor other than A_c, but the shapes will always look like this.

5.a

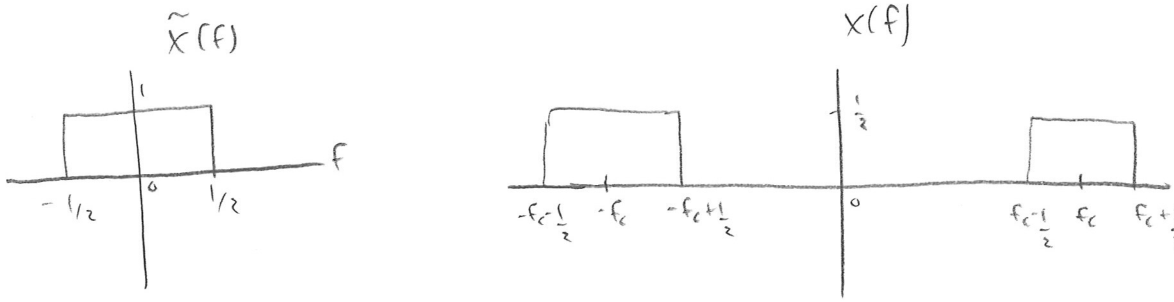
$$\tilde{X}(f) = \mathcal{F}\{\operatorname{sinc}(t)\} = \boxed{\Pi(f)}$$

$$X(f) = \mathcal{F}\{\operatorname{Re}(\tilde{x}e^{2\pi j f_c t})\} = \mathcal{F}\{\tfrac{1}{2}\tilde{x}(t)e^{2\pi j f_c t} + \tfrac{1}{2}\tilde{x}^*(t)e^{-2\pi j f_c t}\}$$

$$= \tfrac{1}{2}\tilde{X}(f-f_c) + \tfrac{1}{2}\tilde{X}^*(-(f+f_c))$$

$$= \boxed{\tfrac{1}{2}\Pi(f-f_c) + \tfrac{1}{2}\Pi(f+f_c)} \quad (\Pi \text{ is even})$$

5.b



HW4

5.c

$$M(f) = \Pi(f)$$

$$\hat{M}(f) = -j\,\mathrm{sgn}(f)\,\Pi(f)$$

$$\tilde{X}(f) = M(f) - j\hat{M}(f) = \Pi(f) - j(-j\,\mathrm{sgn}(f))\Pi(f)$$

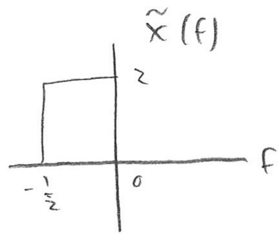
$$= \boxed{\Pi(f)(1-\mathrm{sgn}(f)) = \Pi(f)(2u(-f)) = \begin{cases} 2 & -\frac{1}{2} < f < 0 \\ 0 & \text{else} \end{cases}}$$

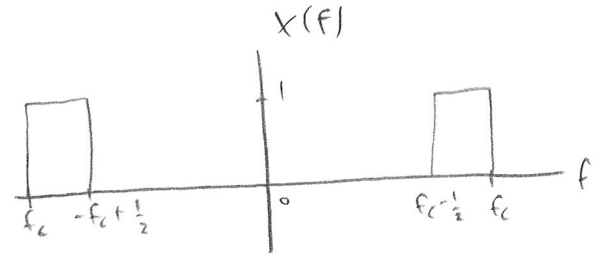
$$X(f) = \tfrac{1}{2}\tilde{X}(f-f_c) + \tfrac{1}{2}\tilde{X}^*(-(f+f_c))$$

$$= \Pi(f-f_c)\,u(-(f-f_c)) + \Pi(f+f_c)\,u(+f+f_c)$$

$$= \begin{cases} 1 & f_c - \frac{1}{2} < f < f_c \\ 1 & -f_c < f < -f_c + \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

5.d





5.e

$$\tilde{X}(f) = M(f) + j\hat{M}(f) = \boxed{\Pi(f)(1+\mathrm{sgn}(f)) = 2u(f)\,\Pi(f) = \begin{cases} 2 & 0 < f < \frac{1}{2} \\ 0 & \text{else} \end{cases}}$$

$$x(f) = \tfrac{1}{2}\tilde{X}(f-f_c) + \tfrac{1}{2}\tilde{X}(-(f+f_c)) = \boxed{\begin{aligned} &\Pi(f-f_c)\,u(f-f_c) + \Pi(f+f_c)\,u(-(f+f_c)) \\ &= \begin{cases} 1 & f_c < f < f_c + \frac{1}{2} \\ 1 & -\frac{1}{2} - f_c < f < f_c \\ 0 & \text{else} \end{cases} \end{aligned}}$$

5.f